On billiard approach in multidimensional cosmological models

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Abstract

A short overview of the billiard approach for cosmological-type models with n Einstein factor-spaces is presented. We start with the billiard representation for pseudo-Euclidean Toda-like systems of cosmological origin. Then we consider cosmological model with multicomponent "perfect-fluid" and cosmological-type model with composite branes. The second one describes cosmological and spherically-symmetric configurations in a theory with scalar fields and fields of forms. The conditions for appearance of asymptotical Kasner-like and oscillating behaviors in the limit $\tau \to +0$ and $\tau \to +\infty$ (where τ is a "synchronous-type" variable) are formulated (e.g. in terms of inequalities on Kasner parameters). Examples of billiards related to the hyperbolic Kac-Moody algebras E_{10}, AE_3 and $A_{1,II}$ are given.

PACS numbers: 04.50.-h.

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1 Introduction

This paper is devoted to the billiard approach for multidimensional cosmological-type models defined on the manifold $(u_-, u_+) \times M_1 \times \ldots \times M_n$, where all M_i are Einstein spaces. This approach was inspired by Chitre's idea [1] of explanation the BLK-oscillations [2] in the mixmaster model (based on Bianchi-IX metric) [3, 4] by using a simple triangle billiard in the Lobachevsky space H^2.

Let us briefly overview the "history" of the billiard approach in multidimensional cosmology. In multidimensional case the billiard representation for cosmological model with multicomponent "perfect" fluid was introduced in [5, 6, 7]. In our paper [7] the finiteness of the billiard volume was formulated in terms of the so-called illumination problem, the inequalities on Kasner parameters were written and the "quantum billiard" was also considered. The mathematical quintessence of the derivation of billiard representation was considered in our paper [8] devoted to pseudo-Euclidean Toda-like systems of cosmological origin.

The billiard approach for multidimensional models with scalar fields and fields of forms (e.g. for supergravitational ones) was suggested in our paper [9]. This paper contains the inequalities on Kasner parameters that played a key role in the proof of Damour and Henneaux conjecture on "chaotic" behavior in superstring-inspired (e.g. supergravitational) models [10] (for more detailed explanation see review article [11]). We note that at the moment there are no examples of cosmological (S-brane) configurations in $D = 11$ supergravity with diagonal (or block-diagonal) metrics that have an oscillating behaviour near the singularity (see [12]).

There was also an important observation made in [13]: for certain superstring-inspired (e.g. supergravitational) models the parts of billiards are related to Weyl chambers of certain hyperbolic Kac-Moody (KM) Lie algebras [14, 15, 16, 17]. This observation drastically simplifies the proof of the finiteness of the billiard volume. Using this approach the old well-known result of Demaret, Henneaux and Spindel [18] on critical dimension of pure gravity was explained using hyperbolic algebras in [19].

It should be noted that earlier few examples of hyperbolic KM algebras were considered in our papers (with co-authors) [20, 21, 22] in a context of exact solutions with branes, see also [23].

Here we overview some our results from [7, 9, 8] with a certain generalization, e.g. we consider two asymptotical regions when i) $\tau \to +0$ (labelled by $\varepsilon = +1$) and ii) $\tau \to +\infty$ ($\varepsilon = -1$) where τ is a "synchronous-type" variable (it may be time variable or radial variable for spherically-symmetric configurations in the model with form fields and scalar fields) and give some concrete examples, related to hyperbolic KM algebras.

2 Billiard representation for pseudo-Euclidean Toda-like systems of cosmological origin

Here we consider a pseudo-Euclidean Toda-like system described by the following Lagrangian

$$L = L(z^a, \dot{z}^a, \mathcal{N}) = \frac{1}{2}\mathcal{N}^{-1}\eta_{ab}\dot{z}^a\dot{z}^b - \mathcal{N}V(z), \tag{2.1}$$

where $\mathcal{N} > 0$ is the Lagrange multiplier (modified lapse function),
$(\eta_{ab}) = \text{diag}(-1, +1, \ldots, +1)$ is the matrix of minisuperspace metric, $a, b = 0, \ldots, N-1$,
and

$$V(z) = \sum_{\alpha=1}^{M} A_\alpha \exp(u_a^\alpha z^a) \tag{2.2}$$

is the potential, all $A_\alpha \neq 0$.

We consider the behavior of the dynamical system (2.1) for $N \geq 3$ in the limit

$$z^2 \equiv -(z^0)^2 + (\vec{z})^2 \to -\infty, \qquad z = (z^0, \vec{z}) \in \mathcal{V}_{-\varepsilon}, \tag{2.3}$$

where $\mathcal{V}_{-\varepsilon} \equiv \{(z^0, \vec{z}) \in \mathbb{R}^N : \varepsilon z^0 < -|\vec{z}|\}$ is the lower or upper light cone for $\varepsilon = +1, -1$ respectively. The limit (2.3) implies $z^0 \to \mp\infty$ for $\varepsilon = \pm 1$. For $\varepsilon = +1$ it describes (under certain additional assumptions imposed) approaching to the singularity in corresponding cosmological models.

We impose the following restrictions on vectors $u^\alpha = (u_0^\alpha, \vec{u}^\alpha)$ in the potential (2.2)

$$1)\ A_\alpha > 0 \text{ if } (u^\alpha)^2 = -(u_0^\alpha)^2 + (\vec{u}^\alpha)^2 > 0; \tag{2.4}$$
$$2)\ \varepsilon u_0^\alpha > 0 \text{ if } (u^\alpha)^2 \leq 0. \tag{2.5}$$

Let us consider the behavior of the dynamical system, described by the Lagrangian (2.1) for $N \geq 3$ in the limit (2.3). We restrict the Lagrange system (2.1) on $\mathcal{V}_{-\varepsilon}$.

Remark 1. In a general case the shifted cone $\mathcal{V}_{-\varepsilon}(\eta) \equiv \{(z^0, \vec{z}) \in \mathbb{R}^N : \varepsilon(z^0 - \eta^0) < -|\vec{z} - \vec{\eta}|\}$ should be considered, where $\eta = (\eta^0, \vec{\eta})$. Here we put $\eta = 0$ for simplicity.

We introduce an analogue of the Misner-Chitre coordinates in $\mathcal{V}_{-\varepsilon}$:

$$z^0 = -\varepsilon \exp(-\varepsilon y^0)\frac{1 + \vec{y}^2}{1 - \vec{y}^2}, \tag{2.6}$$

$$\vec{z} = -2\varepsilon \exp(-\varepsilon y^0)\frac{\vec{y}}{1 - \vec{y}^2}, \tag{2.7}$$

$|\vec{y}| < 1$, and fix the gauge

$$\mathcal{N} = \exp(-2\varepsilon y^0) = -z^2. \tag{2.8}$$

In what follows we consider $(N-1)$-dimensional Lobachevsky space H^{N-1} realized as a unit ball $H^{N-1} = D^{N-1} \equiv \{\vec{y} = (y^1, \ldots, y^{N-1}) : |\vec{y}| < 1\}$ with the metric $h = 4\delta_{ij}(1 - \vec{y}^2)^{-2}dy^i \otimes dy^j$.

The set of indices $\Delta_+ \equiv \{\alpha : (u^\alpha)^2 > 0\}$ defines a billiard B in H^{N-1}:

$$B = \bigcap_{\alpha \in \Delta_+} B(u^\alpha), \qquad (2.9)$$

where the subset $B(u^\alpha)$ consists of points $\vec{y} \in D^{n-1}$ obeying:
  i) $(\vec{y} - \vec{v}^\alpha)^2 > (\vec{v}^\alpha)^2 - 1$ for $\varepsilon u_0^\alpha > 0$;
  ii) $(\vec{y} - \vec{v}^\alpha)^2 < (\vec{v}^\alpha)^2 - 1$ for $\varepsilon u_0^\alpha < 0$;
  iii) $\varepsilon \vec{y} \vec{u}^\alpha > 0$ for $u_0^\alpha = 0$;
$\alpha \in \Delta_+$.
Here

$$\vec{v}^\alpha = -\vec{u}^\alpha / u_0^\alpha \text{ for } u_0^\alpha \neq 0. \qquad (2.10)$$

B is an open domain. Its boundary $\partial B = \bar{B} \backslash B$ is formed by certain parts of $m_+ = |\Delta_+|$ $(N-2)$-dimensional planes or spheres with centers in the points (2.10) ($|\vec{v}^\alpha| > 1$) and radii $r_\alpha = \sqrt{(\vec{v}^\alpha)^2 - 1}$.

It may be shown that in the limit $y^0 \to -\varepsilon\infty$ (or, equivalently, in the limit (2.3)) the Lagrange equations for the Lagrangian (2.1) with the gauge fixing (2.8) (under restrictions (2.4) and (2.5) imposed) are reduced to Lagrange equations for the Lagrangian

$$L_B = \frac{1}{2} h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j - V(\vec{y}, B), \qquad (2.11)$$

where

$$V(\vec{y}, B) \equiv \quad 0, \quad \vec{y} \in B,$$
$$+\infty, \quad \vec{y} \in D^{N-1} \backslash B, \qquad (2.12)$$

is a potential describing m_+ billiard walls. The y^0-variable is separated: $y^0 = \omega(t - t_0)$, ($\omega \neq 0$, t_0 are constants) and the energy constraint $E_B = \omega^2/2$ should be imposed (for $\varepsilon = 1$ see [7, 8]).

We put $\omega > 0$, then the limit $t \to -\varepsilon\infty$ corresponds to (2.3). When $\Delta_+ = \emptyset$, we have $B = D^{N-1}$ and Lagrangian (2.11) describes the geodesic flow on the Lobachevsky space H^{N-1}. In this case there are two families of non-trivial geodesic solutions (i.e. $y(t) \neq const$): lines or semi-circles orthogonal to the boundary S^{N-2} [7].

When $\Delta_+ \neq \emptyset$ Lagrangian (2.11) describes a motion of a particle of unit mass, moving in the billiard B. For cosmological models (see next section) the geodesic motion in B corresponds to a "Kasner epoch" and the reflection from the boundary corresponds to a change of Kasner epoch.

When billiard B has the infinite volume there are open zones at the infinite sphere $|\vec{y}| = 1$. After a finite number of reflections from the boundary the particle (in a general case) moves toward one of these open zones. For the corresponding cosmological model we get the "Kasner-like" asymptotical behavior in the limit $t \to -\varepsilon\infty$.

For (non-empty) billiards with finite volume the motion of the particle describes an "oscillatory-like" asymptotical behaviour of the corresponding cosmological model in the limit $t \to -\varepsilon\infty$.

In [7] we proposed a simple "illumination" criterion for the finiteness of the volume of B, which in the extended form is:

Proposition 1. *The billiard B (2.9) has a finite volume if and only if: the point-like sources of light located at the points $\vec{v}^\alpha$ (2.10) for $\varepsilon u_0^\alpha > 0$, the sources at infinity $-\infty \varepsilon \vec{u}^\alpha$ for $u_0^\alpha = 0$ and "anti-sources" located at points (2.10) for $\varepsilon u_0^\alpha < 0$ illuminate the unit sphere S^{N-2}. For "anti-source" the shadowed domain coincides with the illuminated domain for the usual source located at the same point (and vice versa).*

This proposition was proved in [7] for usual sources of light, when all $u_0^\alpha > 0$ and $\varepsilon = +1$.

The problem of illumination of a convex body in a vector space by point-like sources for the first time was considered in [24, 25]. For the case of S^{N-2} this problem is equivalent to the problem of covering the spheres with spheres [26, 27]. There exists a topological bound on the number of usual point-like sources m_+ illuminating sphere S^{N-2} [25]:

$$m_+ \geq N. \tag{2.13}$$

3 Billiard representation for a cosmological model with m-component perfect-fluid

In this section we consider a cosmological model describing the evolution of n Einstein spaces in the presence of m-component perfect-fluid matter. The metric of the model

$$g = -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{n} \exp[2x^i(t)]\hat{g}^i, \tag{3.1}$$

is defined on the manifold

$$M = (t_-, t_+) \times M_1 \times \ldots \times M_n, \tag{3.2}$$

where the manifold M_i with the metric g^i is an Einstein space of dimension d_i, $R_{m_i n_i}[g^i] = \xi_i g^i_{m_i n_i}$, $i = 1, \ldots, n$; $n \geq 2$. Here and in what follows $\hat{g}^i = p_i^* g^i$ is the pullback of the metric g^i to the manifold M by the canonical projection: $p_i : M \to M_i$, $i = 1, \ldots, n$.

The energy-momentum tensor is adopted in the following form

$$T_N^M = \sum_{\alpha=1}^{m} T_N^{M(\alpha)}, \tag{3.3}$$

$$(T_N^{M(\alpha)}) = \text{diag}(-\rho^{(\alpha)}(t), p_1^{(\alpha)}(t)\delta_{k_1}^{m_1}, \ldots, p_n^{(\alpha)}(t)\delta_{k_n}^{m_n}), \tag{3.4}$$

$\alpha = 1, \ldots, m$, with the conservation law constraints imposed:

$$\nabla_M T_N^{M(\alpha)} = 0 \tag{3.5}$$

$\alpha = 1, \ldots, m - 1$.

The Einstein equations

$$R_N^M - \frac{1}{2}\delta_N^M R = \kappa^2 T_N^M \tag{3.6}$$

(κ^2 is the multidimensional gravitational constant) imply $\nabla_M T_N^M = 0$ and consequently $\nabla_M T_N^{M(m)} = 0$.

We suppose that for any α-th component of matter pressures in all spaces are proportional to a density

$$p_i^{(\alpha)}(t) = \left(1 - \frac{u_i^{(\alpha)}}{d_i}\right)\rho^{(\alpha)}(t), \tag{3.7}$$

where $u_i^{(\alpha)}$ are constants, $i = 1, \ldots, n$; $\alpha = 1, \ldots, m$.

The conservation law constraint (3.5) reads $\dot{\rho}^{(\alpha)} + \sum_{i=1}^{n} d_i \dot{x}^i (\rho^{(\alpha)} + p_i^{(\alpha)}) = 0$ and hence using (3.7) we get

$$\rho^{(\alpha)} = A^{(\alpha)} \exp[-2d_i x^i + u_i^{(\alpha)} x^i], \tag{3.8}$$

where $A^{(\alpha)}$ are constant numbers.

It was shown in [28, 7] that the Einstein equations (3.6) for the metric (3.1) and the energy-momentum tensor (3.3), (3.4) with (3.7) are equivalent to the Lagrange equations for the following degenerate Lagrangian

$$L = \frac{1}{2}\exp(-\gamma + \gamma_0(x))G_{ij}\dot{x}^i\dot{x}^j - \exp(\gamma - \gamma_0(x))V(x), \tag{3.9}$$

where

$$\gamma_0 \equiv \sum_{i=1}^{n} d_i x^i, \tag{3.10}$$

and

$$G_{ij} = d_i\delta_{ij} - d_i d_j \tag{3.11}$$

are components of minisuperspace metric [29], and

$$V = V(x) = -\frac{1}{2}\sum_{i=1}^{n} \xi_i d_i \exp(-2x^i + 2\gamma_0(x)) + \sum_{\alpha=1}^{m} \kappa^2 A^{(\alpha)} \exp(u_i^{(\alpha)} x^i) \tag{3.12}$$

is the potential.

The relation (3.12) may be also presented in the form

$$V = \sum_{\alpha=1}^{\bar{M}} A_\alpha \exp(u_i^{(\alpha)} x^i), \tag{3.13}$$

where $\bar{M} = m + n$; $A_\alpha = \kappa^2 A^{(\alpha)}$, $\alpha = 1, \ldots, m$; $A_{m+i} = -\frac{1}{2}\xi_i d_i$ and

$$u_j^{(m+i)} = 2(-\delta_j^i + d_j), \tag{3.14}$$

$i, j = 1, \ldots, n$.

Diagonalization.

The minisuperspace metric $G = G_{ij}dx^i \otimes dx^i$ has the pseudo-Euclidean signature $(-, +, \ldots, +)$ [29], i.e. there exists a linear transformation

$$z^a = e_i^a x^i, \tag{3.15}$$

diagonalizing the minisuperpace metric: $G = \eta_{ab}dz^a \otimes dz^b$ where

$(\eta_{ab}) = (\eta^{ab}) \equiv \mathrm{diag}(-1, +1, \ldots, +1)$, $a, b = 0, \ldots, n - 1$.

Like in [29] we put

$$z^0 = e_i^0 x^i = q^{-1} d_i x^i, \qquad q = [(D-1)/(D-2)]^{1/2}. \tag{3.16}$$

For the volume scale factor $v = \exp(\sum_{i=1}^n d_i x^i) = \exp(qz^0)$ we get $v \to +0$ for $z^0 \to -\infty$ and $v \to +\infty$ for $z^0 \to +\infty$.

Let us denote

$$u_a^\alpha = e_a^i u_i^{(\alpha)}, \tag{3.17}$$

$a = 0, \ldots, n - 1$, where $(e_a^i) = (e_i^a)^{-1}$.

Then the Lagrangian (3.9) written in z-variables is coinciding with the Lagrangian (2.1) with $N = n$, $M \le n + m$ and $\mathcal{N} = \exp(-\gamma_0 + \gamma) > 0$.

It was shown in [7] that

$$u_0^\alpha = \left(\sum_{i=1}^n u_i^{(\alpha)} \right) / q(D-2). \tag{3.18}$$

and

$$(u^\alpha)^2 = (u^{(\alpha)}, u^{(\alpha)}), \tag{3.19}$$

where $(u, v) = G^{ij} u_i v_j$,

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D} \tag{3.20}$$

are components of the matrix inverse to (G_{ij}), $D = 1 + \sum_{i=1}^n d_i$ is the dimension of the manifold (3.2.)

For the curvature u-vectors we get

$$u_0^{m+j} = 2/q > 0, \tag{3.21}$$

and

$$(u^{m+j})^2 = 4 \left(\frac{1}{d_j} - 1 \right) < 0, \tag{3.22}$$

for $d_j > 1$, $j = 1, \ldots, n$. For $d_j = 1$ we have $\xi_j = A_{m+j} = 0$.

Billiard restrictions.

The restrictions 1) and 2) (see (2.4) and (2.5)) read (due to (3.18)-(3.22)):

$$1) \ A^{(\alpha)} > 0 \ \text{if} \ (u^{(\alpha)}, u^{(\alpha)}) > 0; \tag{3.23}$$

$$2a) \ \varepsilon \sum_{i=1}^n u_i^{(\alpha)} > 0 \ \text{if} \ (u^{(\alpha)}, u^{(\alpha)}) \le 0; \tag{3.24}$$

$$2b) \ \xi_j = 0, \ j = 1, \ldots, n, \ \text{if} \ \varepsilon = -1. \tag{3.25}$$

The last condition means that all factor spaces M_i should be Ricci-flat when the case $\varepsilon = -1$ is studied.

Kasner-like parametrization.

Let the billiard has an infinite volume and hence there are open zones at infinity. It may be shown (along a line as it was done in [7] when all $u_0^\alpha > 0$ and $\varepsilon = +1$) that the geodesic motion in H^{n-1} towards one of these zones corresponds to "Kasner-like" asymptotical behaviour of the metric (3.1) in the limit when $\tau \to +0$ for $\varepsilon = +1$ or $\tau \to +\infty$ for $\varepsilon = -1$. Here τ is the synchronous time variable. The asymptotical form of the metric reads

$$g_{as} = -d\tau \otimes d\tau + \sum_{i=1}^{n} A_i \tau^{2\alpha^i} \hat{g}^i, \tag{3.26}$$

$$\sum_{i=1}^{n} d_i \alpha^i = \sum_{i=1}^{n} d_i (\alpha^i)^2 = 1, \tag{3.27}$$

where $A_i > 0$ are constants. Here the Kasner parameters obey the following inequalities:

$$\sum_{i=1}^{n} \varepsilon u_i^{(\nu)} \alpha^i > 0, \tag{3.28}$$

$\nu \in \Delta_+$.

The Kasner set $\alpha = (\alpha^i)$ is in one-to-one correspondence with the unit vector $\vec{n} \in S^{n-2}$: $\alpha^i = e_a^i n^a / q$, $(n^a) = (1, \vec{n})$.

The criterion of the finiteness of the billiard volume (see Proposition 1) may be reformulated in terms of inequalities on the Kasner-like parameters.

Proposition 2. *The (non-empty) billiard B (2.9) has a finite volume if and only if the set of relations (3.27), (3.28) is inconsistent.*

This proposition may be proved along a line as it was done in [7] when all $u_0^\alpha > 0$ and $\varepsilon = +1$. For finite (non-zero) billiard volume we get a never ending asymptotical oscillating behaviour.

Remark 2. Let all factor spaces are Ricci-flat, i.e. $\xi_j = 0$, $j = 1, \ldots, n$. It is not difficult to verify that for a fixed diagonalization procedure (3.15)-(3.17) the billiard B (2.9) is unchanged when the following transformation of parameters is performed:

$$u_i^{(\alpha)} \mapsto -u_i^{(\alpha)}, \qquad \varepsilon \mapsto -\varepsilon, \tag{3.29}$$

$i = 1, \ldots, n$, $\alpha = 1, \ldots, m$. In terms of the parameters $w_i^{(\alpha)} = 1 - \frac{u_i^{(\alpha)}}{d_i}$ (i.e. $p_i^{(\alpha)} = w_i^{(\alpha)} \rho^{(\alpha)}$), the first formula in relation (3.29) reads: $w_i^{(\alpha)} \mapsto \hat{w}_i^{(\alpha)} = 2 - w_i^{(\alpha)}$.

Thus, for a given billiard B describing the behaviour near the singularity (either Kasner-like or never-ending oscillating one) as $\tau \to +0$ ($\varepsilon = +1$) we get the same billiard B for $\tau \to +\infty$ ($\varepsilon = -1$) when the u-parameters are replaced according to (3.29).

Remark 3. For a fixed diagonalization procedure (3.15)-(3.17) the billiard B from Section 2 is unchanged when the following transformation of parameters is done:

$$u_i^{(\alpha)} \mapsto \lambda_{(\alpha)} u_i^{(\alpha)}, \tag{3.30}$$

where $\lambda_{(\alpha)} > 0$, $i = 1, \ldots, n$, $\alpha = 1, \ldots, m$. Here ε is unchanged.

In terms of the w-parameters the relation (3.30) reads $w_i^{(\alpha)} \mapsto \hat{w}_i^{(\alpha)} = \lambda_{(\alpha)}(w_i^{(\alpha)} - 1) + 1$.

Collision formula.

Let the billiard B has a finite volume. In this case we get a never ending oscillation behaviour in the asymptotical regime. In a period between two collisions with potential walls we have a Kasner-like relations for the metric (3.26) with α-parameters obeying (3.27). It may be shown (along the line as it was done in [30] for S-brane solutions) that the set of Kasner parameters $(\alpha^{'i})$ after the collision with the s-th wall (corresponding to the s-th component), $s \in \Delta_+$, is defined by the Kasner set before the collision (α^i) according to the following formula

$$
\alpha^{'i} = \frac{\alpha^i - 2u^{(s)}(\alpha)u^{(s)i}(u^{(s)}, u^{(s)})^{-1}}{1 - 2u^{(s)}(\alpha)(u^{(s)}, u^\Lambda)(u^{(s)}, u^{(s)})^{-1}}, \tag{3.31}
$$

$i = 1, \ldots, n$. Here $u^{(s)}(\alpha) = u_i^{(s)}\alpha^i$, $u^{(s)i} = G^{ij}u_j^{(s)}$ and $u_i^\Lambda = 2d_i$.

4 Billiard representation for a cosmological-type model with branes

Now, we consider the model governed by the action

$$
S = \int_M d^D z \sqrt{|g|}\{R[g] - 2\Lambda - h_{\alpha\beta}g^{MN}\partial_M\varphi^\alpha\partial_N\varphi^\beta \tag{4.1}
$$
$$
- \sum_{a\in\Delta}\frac{\theta_a}{n_a!}\exp[2\lambda_a(\varphi)](F^a)^2_g\},
$$

where $g = g_{MN}dz^M \otimes dz^N$ is a metric on the manifold M, $\dim M = D$, $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$ is a vector from dilatonic scalar fields, $(h_{\alpha\beta})$ is a positive-definite symmetric $l \times l$ matrix ($l \in \mathbb{N}$), $\theta_a \neq 0$, $F^a = dA^a = \frac{1}{n_a!}F^a_{M_1\ldots M_{n_a}}dz^{M_1} \wedge \ldots \wedge dz^{M_{n_a}}$ is a n_a-form ($n_a \geq 2$) on a D-dimensional manifold M, Λ is a cosmological constant and λ_a is a 1-form on $\mathbb{R}^l$: $\lambda_a(\varphi) = \lambda_{a\alpha}\varphi^\alpha$, $a \in \Delta$, $\alpha = 1, \ldots, l$. In (4.1) we denote $|g| = |\det(g_{MN})|$, $(F^a)^2_g = F^a_{M_1\ldots M_{n_a}}F^a_{N_1\ldots N_{n_a}}g^{M_1N_1}\ldots g^{M_{n_a}N_{n_a}}$, $a \in \Delta$, where Δ is some finite set. In models with one time all $\theta_a = 1$ when the signature of the metric is $(-1, +1, \ldots, +1)$.

We consider the manifold

$$
M = (u_-, u_+) \times M_1 \times \ldots \times M_n, \tag{4.2}
$$

with the metric

$$
g = we^{2\gamma(u)}du \otimes du + \sum_{i=1}^n e^{2x^i(u)}\hat{g}^i, \tag{4.3}
$$

where $w = \pm 1$, $g^i = g^i_{m_in_i}(y_i)dy_i^{m_i} \otimes dy_i^{n_i}$ is an Einstein metric on M_i satisfying $R_{m_in_i}[g^i] = \xi_i g^i_{m_in_i}$, $m_i, n_i = 1, \ldots, d_i$; $\xi_i = \text{const}$, $i = 1, \ldots, n$. The functions $\gamma, x^i : (u_-, u_+) \to \mathbb{R}$ are smooth. We denote $d_i = \dim M_i$; $i = 1, \ldots, n$; $D = 1 + \sum_{i=1}^n d_i$. Here u is a variable by convention called "time".

We consider any manifold M_i to be oriented and connected. Then the volume d_i-form

$$
\tau_i \equiv \sqrt{|g^i(y_i)|}\ dy_i^1 \wedge \ldots \wedge dy_i^{d_i}, \tag{4.4}
$$

and signature parameter

$$\varepsilon(i) \equiv \text{sign}(\det(g^i_{m_i n_i})) = \pm 1 \qquad (4.5)$$

are correctly defined for all $i = 1, \ldots, n$.

Let $\Omega = \Omega(n)$ be a set of all non-empty subsets of $\{1, \ldots, n\}$. The number of elements in Ω is $|\Omega| = 2^n - 1$. For any $I = \{i_1, \ldots, i_k\} \in \Omega$, $i_1 < \ldots < i_k$, we denote

$$\tau(I) \equiv \hat{\tau}_{i_1} \wedge \ldots \wedge \hat{\tau}_{i_k}, \qquad (4.6)$$
$$\varepsilon(I) \equiv \varepsilon(i_1) \ldots \varepsilon(i_k), \qquad (4.7)$$
$$d(I) \equiv \sum_{i \in I} d_i. \qquad (4.8)$$

Here $\hat{\tau}_i = p_i^* \hat{\tau}_i$ is the pullback of the form τ_i to the manifold M by the canonical projection: $p_i : M \to M_i$, $i = 1, \ldots, n$. We also put $\tau(\emptyset) = \varepsilon(\emptyset) = 1$ and $d(\emptyset) = 0$.

For fields of forms we consider the following composite electromagnetic ansatz

$$F^a = \sum_{I \in \Omega_{a,e}} \mathcal{F}^{(a,e,I)} + \sum_{J \in \Omega_{a,m}} \mathcal{F}^{(a,m,J)} \qquad (4.9)$$

where

$$\mathcal{F}^{(a,e,I)} = d\Phi^{(a,e,I)} \wedge \tau(I), \qquad (4.10)$$
$$\mathcal{F}^{(a,m,J)} = e^{-2\lambda_a(\varphi)} * (d\Phi^{(a,m,J)} \wedge \tau(J)) \qquad (4.11)$$

are elementary forms of electric and magnetic types respectively, $a \in \Delta$, $I \in \Omega_{a,e}$, $J \in \Omega_{a,m}$ and $\Omega_{a,v} \subset \Omega$, $v = e, m$. In (4.11) $* = *[g]$ is the Hodge operator on (M, g).

For scalar functions we put

$$\varphi^\alpha = \varphi^\alpha(u), \quad \Phi^s = \Phi^s(u), \qquad (4.12)$$

$s \in S$.

Here and below

$$S = S_e \sqcup S_m, \quad S_v = \sqcup_{a \in \Delta} \{a\} \times \{v\} \times \Omega_{a,v}, \qquad (4.13)$$

$v = e, m$ ($\sqcup$ means the union of non-intersecting sets). The set S consists of elements $s = (a_s, v_s, I_s)$, where $a_s \in \Delta$ is the colour index, $v_s = e, m$ is the electro-magnetic index and set $I_s \in \Omega_{a_s, v_s}$ describes the location of a brane.

Due to (4.10) and (4.11) $d(I) = n_a - 1$, $d(J) = D - n_a - 1$, for $I \in \Omega_{a,e}$ and $J \in \Omega_{a,m}$ (i.e. in electric and magnetic cases, respectively).

Restrictions on brane intersections. Here we put two restrictions on sets of branes that guarantee the block-diagonal form of the energy-momentum tensor and the existence of the sigma-model representation (without additional constraints):

$$(\mathbf{R1}) \quad d(I \cap J) \le d(I) - 2, \qquad (4.14)$$

for any $I, J \in \Omega_{a,v}$, $a \in \Delta$, $v = e, m$ (here $d(I) = d(J)$) and

$$\textbf{(R2)} \quad d(I \cap J) \neq 0, \tag{4.15}$$

for any $I \in \Omega_{a,e}$, $J \in \Omega_{a,m}$, $a \in \Delta$.

It follows from [31] that equations of motion for the model (4.1) and the Bianchi identities: $d\mathcal{F}^s = 0$, $s \in S_m$, for fields from (4.3), (4.9)-(4.12), when Restrictions (4.14) and (4.15) are imposed, are equivalent to equations of motion for the 1-dimensional σ-model with the action

$$S_\sigma = \int du \mathcal{N}^{-1} \Big\{ \hat{G}_{AB} \dot{\sigma}^A \dot{\sigma}^B + \sum_{s \in S} \varepsilon_s \exp[-2U^s(\sigma)](\dot{\Phi}^s)^2 - 2\mathcal{N}^2 V_w \Big\}, \tag{4.16}$$

where $\dot{X} \equiv dX/du$,

$$V_w = -w\Lambda e^{2\gamma_0(x)} + \frac{w}{2} \sum_{i=1}^{n} \xi_i d_i e^{-2x^i + 2\gamma_0(x)} \tag{4.17}$$

is the potential, $\gamma_0(x) \equiv \sum_{i=1}^{n} d_i x^i$ and $\mathcal{N} = \exp(-\gamma_0 + \gamma) > 0$.

In (4.17) $(\sigma^A) = (x^i, \varphi^\alpha)$, the index set S is defined in (4.13),

$$(\hat{G}_{AB}) = \text{diag}(G_{ij}, h_{\alpha\beta}) \tag{4.18}$$

is the matrix of a minisuperspace metric and

$$U^s(\sigma) = U^s_A \sigma^A = \sum_{i \in I_s} d_i x^i - \chi_s \lambda_{a_s}(\varphi), \quad (U^s_A) = (d_i \delta^i_{I_s}, -\chi_s \lambda_{a_s \alpha}) \tag{4.19}$$

are the so-called U-(co)vectors, $s = (a_s, v_s, I_s)$. Here $\chi_e = +1$ and $\chi_m = -1$;

$$\delta^i_I = \sum_{j \in I} \delta^i_j \tag{4.20}$$

is an indicator of i belonging to I: $\delta^i_I = 1$ for $i \in I$ and $\delta_{iI} = 0$ otherwise; and

$$\varepsilon_s = \varepsilon(I_s)\theta_{a_s} \text{ for } v_s = e; \qquad \varepsilon_s = -\varepsilon[g]\varepsilon(I_s)\theta_{a_s} \text{ for } v_s = m, \tag{4.21}$$

$s \in S$, and $\varepsilon[g] \equiv \text{sign} \det(g_{MN})$.

Now we integrate the Lagrange equations corresponding to Φ^s (i.e. the "Maxwell equations" for $s \in S_e$ and Bianchi identities for $s \in S_m$):

$$\frac{d}{du} \Big(\mathcal{N}^{-1} \exp(-2U^s(\sigma))\dot{\Phi}^s \Big) = 0 \iff \dot{\Phi}^s = Q_s \mathcal{N} \exp(2U^s(\sigma)), \tag{4.22}$$

where Q_s are constants, $s \in S$. We put $Q_s \neq 0$ for all $s \in S$. For fixed $Q = (Q_s, s \in S)$ the Euler-Lagrange equations for the action (4.16) corresponding to $(\sigma^A) = (x^i, \varphi^\alpha)$, when equations (4.22) are substituted, are equivalent to Lagrange equations for the Lagrangian

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$$L = \mathcal{N}^{-1}\hat{G}_{AB}\dot{\sigma}^A\dot{\sigma}^B - \mathcal{N}V, \tag{4.23}$$

where

$$V = V_w + \frac{1}{2}\sum_{s\in S}\varepsilon_s Q_s^2 \exp[2U^s(\sigma)] \tag{4.24}$$

and the matrix $(\hat{G}_{AB})$ is defined in (4.18). This potential may be rewritten as

$$V = \sum_{r\in S_*} A_r \exp[2U^r(\sigma)], \tag{4.25}$$

where $S_* = S \sqcup \{\Lambda\} \sqcup \{1,\ldots,n\}$, $A_s = \frac{1}{2}\varepsilon_s Q_s^2$, $s \in S$, $A_\Lambda = -w\Lambda$ and $A_i = \frac{w}{2}\xi_i d_i$, $i = 1,\ldots,n$. Here $U_i^\Lambda = d_i$, $U_j^i = -\delta_j^i + d_j$ and all other components are zero.

We remind that [31]

$$(U^\Lambda, U^\Lambda) = -q^2 < 0, \qquad (U^j, U^j) = \frac{1}{d_j} - 1 < 0, \tag{4.26}$$

for $d_j < 1$ and

$$(U^s, U^s) = d(I_s)\left(1 + \frac{d(I_s)}{2-D}\right) + \lambda_{a_s\alpha}\lambda_{a_s\beta}h^{\alpha\beta}. \tag{4.27}$$

Here and in what follows
$$(U, U') = \hat{G}^{AB}U_A U'_B, \tag{4.28}$$
where the matrix $(\hat{G}^{AB}) = \mathrm{diag}(G^{ij}, h^{\alpha\beta})$ is inverse to $(\hat{G}_{AB})$ and $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$.

Diagonalization.

The minisuperspace metric $\hat{G} = \hat{G}_{AB}d\sigma^A \otimes d\sigma^B$ has the pseudo-Euclidean signature $(-,+,\ldots,+)$ (since $(h_{\alpha\beta})$ is a positive-definite), i.e. there exists a linear transformation

$$z^a = e_A^a\sigma^A, \tag{4.29}$$

diagonalizing the minisuperpace metric: $\hat{G} = \eta_{ab}dz^a \otimes dz^b$ where $(\eta_{ab}) = (\eta^{ab}) \equiv diag(-1,+1,\ldots,+1)$, $a,b = 0,\ldots,N-1$; $N = n+l$.

Like in the previous section we put

$$z^0 = e_A^0\sigma^A = q^{-1}d_i x^i. \tag{4.30}$$

Let us denote
$$\hat{U}_a^r = e_a^A U_A^r \tag{4.31}$$
$a = 0,\ldots,N-1$, where $(e_a^A) = (e_A^a)^{-1}$. We get

$$\eta^{ab}\hat{U}_a^r\hat{U}_b^{r'} = (U^r, U^{r'}) \tag{4.32}$$

for all r, r'.

Then, the Lagrangian (4.23) written in z-variables is coinciding (after a suitable redefinitions of parameters e.g. $2\hat{U}_a^r = u_a^r$, and indices) with Lagrangian (2.1) where $N = n+l$ and $M \leq |S| + n + 1$.

In what follows we will use the following inequalities [9]

$$\hat{U}_0^\Lambda = q > 0, \qquad \hat{U}_0^j = 1/q > 0, \tag{4.33}$$

$j = 1, \ldots, n$, and

$$\hat{U}_0^s = d(I_s)/\sqrt{(D-2)(D-1)} > 0, \tag{4.34}$$

$s \in S$.

Billiard restrictions.

For U-vectors the restrictions 1) and 2) (see (2.4), (2.5)), imply (due to (4.26) and (4.32)-(4.34))

$$\begin{array}{lll} 1) & \varepsilon_s > 0 \text{ for } (U^s, U^s) > 0; & (4.35) \\ 2a) & \text{all } (U^s, U^s) > 0 \text{ if } \varepsilon = -1; & (4.36) \\ 2b) & \xi^j = \Lambda = 0 \text{ if } \varepsilon = -1; & (4.37) \end{array}$$

$s \in S$, $j = 1, \ldots, n$.

Remark 4. For $\theta_a = 1$, $a \in \Delta$, and $\varepsilon[g] = -1$, the inequality $\varepsilon_s > 0$ means that all $\varepsilon(I_s) = 1$, i.e. a brane with positive (U^s, U^s) should have either Euclidean worldvolume or that containing even number of "times". The inequality $(U^s, U^s) > 0$ is satisfied in a special case when $d(I_s) < D - 2$.

Let $S_+ = \{s \in S : (U^s, U^s) > 0\}$. In this model the branes with $s \in S_+$ are the only matter components responsible for asymptotical formation of billiard walls (when $\varepsilon = -1$ we should put $S = S_+$).

Kasner-like solutions and oscillating behaviour.

Let a billiard B has an infinite volume and hence there are open zones at infinity.

Then we get "Kasner-like" asymptotical behaviour of the metric and scalar fields in the limit when $\tau \to +0$ (for $\varepsilon = +1$) or $\tau \to +\infty$ (for $\varepsilon = -1$):

$$g = w d\tau \otimes d\tau + \sum_{i=1}^n A_i \tau^{2\alpha^i} \hat{g}^i, \tag{4.38}$$

$$\varphi^\beta = \alpha^\beta \ln \tau + \varphi_0^\beta, \tag{4.39}$$

$$\sum_{i=1}^n d_i \alpha^i = \sum_{i=1}^n d_i (\alpha^i)^2 + \alpha^\beta \alpha^\gamma h_{\beta\gamma} = 1, \tag{4.40}$$

where $w = \pm 1$, $A_i > 0$, φ_0^β are constants $i = 1, \ldots, n$; $\beta, \gamma = 1, \ldots, l$. The the set of Kasner parameters $\alpha = (\alpha^A) = (\alpha^i, \alpha^\gamma)$ obeys the relations

$$\varepsilon U^s(\alpha) = \varepsilon U_A^s \alpha^A = \varepsilon \left(\sum_{i \in I_s} d_i \alpha^i - \chi_s \lambda_{a_s \gamma} \alpha^\gamma \right) > 0, \tag{4.41}$$

$s \in S_+$. Thus, we get $U^s(\alpha) > 0$ for $\tau \to +0$ and $U^s(\alpha) < 0$ for $\tau \to +\infty$, $s \in S_+$.

Here τ is the "synchronous time" variable. The set of Kasner parameters α is in one-to-one correspondence with the unit vector $\vec{n} \in S^{N-2}$: $\alpha^A = e_a^A n^a/q$, where $(n^a) = (1, \vec{n})$.

Proposition 3. *The (non-empty) billiard B (2.9) has a finite volume if and only if there are no α satisfying the relations (4.40) and (4.41).*

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This proposition may be proved just along the line as it was done in [9] for the case $\varepsilon = +1$ when all $d(I_s) < D - 2$ and all $\varepsilon_s = +1$. We remind that for finite (non-zero) billiard volume we get a never ending asymptotical oscillating behaviour.

Collision formula and scattering law.

It was shown in [30] that the set of Kasner parameters (α'^A) after the collision with the s-th wall is defined by the Kasner set before the collision (α^A) according to the following formula

$$\alpha'^A = \frac{\alpha^A - 2U^s(\alpha)U^{sA}(U^s, U^s)^{-1}}{1 - 2U^s(\alpha)(U^s, U^\Lambda)(U^s, U^s)^{-1}}. \tag{4.42}$$

Here U^s is a brane co-vector corresponding to the s-th wall and $U^{sA} = \hat{G}^{AB}U^s_B$. In the special case of one scalar field and 1-dimensional factor-spaces (i.e. $l = d_i = 1$) this formula was suggested earlier in [10]. Another special case of the collision law for multidimensional multi-scalar cosmological model with exponential potentials was considered in [32].

Recently in [33] the exact S-brane solution (either electric or magnetic) in a model with l scalar fields and one antisymmetric form of rank $m \geq 2$ was considered. All factor spaces $M_1, ..., M_n$ were supposed to be Ricci-flat and $\Lambda = 0$. A special solution governed by the function *cosh* was singled out. It was shown that this special solution has Kasner-like asymptotics in the limits $\tau \to +0$ and $\tau \to +\infty$, where τ is the synchronous time variable. A relation between two sets of Kasner parameters α_∞ and α_0 was found. Remarkably, this relation, named as "scattering law" formula, coincided with the "collision law" formula (4.42).

5 Examples of billiards related to hyperbolic Kac-Moody algebras

The special class of billiards with finite volumes occurs in the model (4.1) when

$$2\frac{(U^s, U^{s'})}{(U^{s'}, U^{s'})} = A_{ss'}, \tag{5.1}$$

$s, s' \in S_+$, where $A = (A_{ss'})$ is the (generalized) Cartan matrix for hyperbolic Lorentzian Kac-Moody (KM) algebra $\mathcal{G}$, see [14, 16]. In this case the billiard is a projection of the Weyl chamber on the Lobachevsky space H^{N-1} ($|S_+| = N$ is the rank of $\mathcal{G}$). Here [31]

$$(U^s, U^{s'}) = d(I_s \cap I_{s'}) + \frac{d(I_s)d(I_{s'})}{2 - D} + \chi_s\chi_{s'}\lambda_{a_s\alpha}\lambda_{a_{s'}\beta}h^{\alpha\beta}. \tag{5.2}$$

We remind that hyperbolic KM algebras are by definition Lorentzian Kac-Moody algebras with the property that removing any node from their Dynkin diagram leaves one with a Dynkin diagram of the affine or finite type. For exact solutions with branes corresponding non-singular KM algebras (e.g. hyperbolic ones) see [23].

Sometimes the billiard B may be cut into several identical (isomorphic) parts B_k so that any B_k corresponds to the hyperbolic algebra $\mathcal{G}$. Then the volume of B is finite.

Example 1. Let us consider model (4.1) with $D = 11$, $l = 0$ (scalar fields are absent) and F^I are 4-forms ($a = I$), $I \in \Delta = \{I \in \{1, \ldots, 10\} : |I| = 3\}$. Thus, Δ contains all

subsets of $\{1, \ldots, 10\}$ having 3 elements. The number of such forms is 120. We consider the non-composite electric S-brane ansatz when all $d_i = 1$. In this case the billiard B for $\varepsilon = +1$ belongs to Lobachevsky space H^9. It may be cut on several identical parts B_k (using the so-called symmetry walls) such that any B_k corresponds to the hyperbolic algebra E_{10} with the Dynkin diagram pictured on Fig. 1. The volume of B is finite. This billiard appeared for composite electric S-brane configuration with non-diagonal metric in $D = 11$ supergravity [10, 13].
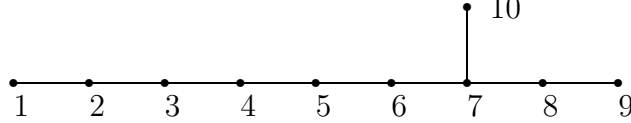

Fig. 1. *Dynkin diagram for E_{10} hyperbolic KM algebra*

For the model with multicomponent perfect fluid an analogous relation

$$2\frac{(u^{(s)}, u^{(s')})}{(u^{(s')}, u^{(s')})} = A_{ss'}, \tag{5.3}$$

$s, s' \in \Delta_+$, gives us an example of billiard with finite volume corresponding to the hyperbolic KM algebra $\mathcal{G}$ (with the Cartan matrix $A = (A_{ss'})$).

Example 2. The billiard with a finite volume corresponding to the hyperbolic algebra E_{10} occurs for $D = 11$ in the cosmological model with ten-component perfect fluid and 1-dimensional factor spaces ($d_i = 1$) when the fluid u-vectors are the following: $u_i^{(j)} = \lambda\varepsilon(\delta_i^j - \delta_i^{j+1})$ for $j = 1, \ldots, 9$, and $u_i^{(10)} = \lambda\varepsilon(\delta_i^8 + \delta_i^9 + \delta_i^{10})$, $i = 1, \ldots, 10$, where $\lambda > 0$ and $\varepsilon = \pm 1$. This 10-component anisotropic fluid model with equations of state parametrized by $\lambda > 0$ and $\varepsilon = \pm 1$ leads to the oscillating behaviour of scale factors for $\tau \to +0$ if $\varepsilon = +1$ and for $\tau \to +\infty$ if $\varepsilon = -1$.

Example 3. The billiard (with a finite volume) corresponding to the hyperbolic KM algebra (that is number 7 in classification of [17] and $A_{1,II}$ in classification of [15])) with the Cartan matrix

$$(A_{ss'}) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} \tag{5.4}$$

occurs in $D = 4$ for the cosmological model with 3-component perfect fluid and 1-dimensional factor spaces ($d_i = 1$) when the fluid u-vectors are the following: $u_i^{(j)} = 2\lambda\varepsilon\delta_i^j$, for $i, j = 1, 2, 3$, where $\lambda > 0$ and $\varepsilon = \pm 1$. This billiard is coinciding with the Chitre's billiard for Bianchi-IX model ($\varepsilon = +1, \lambda = 2$). See Fig. 4 in [8].

Example 4. Another example of a billiard (with a finite volume) corresponding to the hyperbolic KM algebra $AE_3 = \mathcal{F}_3$ ($A_{1,0}$ in classification of [15]) with the Cartan matrix

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \tag{5.5}$$

occurs in the 4-dimensional cosmological model with 3-component perfect fluid and 1-dimensional factor spaces ($d_i = 1$) when the fluid u-vectors have the following form:

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$u_i^{(j)} = \lambda\varepsilon(\delta_i^j - \delta_i^{j+1})$, for $j = 1, 2$, and $u_i^{(3)} = 2\lambda\varepsilon\delta_i^3$, $i = 1, 2, 3$, where $\lambda > 0$ and $\varepsilon = \pm 1$. This billiard may be obtained from that for Bianchi-IX model by cutting it into six identical parts (using three lines crossing the center).

6 Conclusions

Here we reviewed the billiard approach for cosmological-type models with n Einstein factor-spaces. First, we have considered a derivation of the billiard approach for pseudo-Euclidean Toda-like systems of cosmological origin. Then we have applied the billiard scheme to the cosmological model with multicomponent "perfect-fluid" and to cosmological-type model with composite branes. We have also formulated the conditions for appearance of asymptotical Kasner-like behaviour and "never ending" oscillating behavior in the limit $\tau \to +0$ and $\tau \to +\infty$ (where τ is the "synchronous-type" variable) in terms of inequalities on Kasner parameters. We have also suggested examples of billiards related to the hyperbolic Kac-Moody algebras E_{10}, AE_3 and $A_{1,II}$.

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